A relational solution to the problem of time in quantum mechanics and quantum gravity induces a fundamental mechanism for quantum decoherence

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The use of a relational time in quantum mechanics is a framework in which one promotes to quantum operators all variables in a system, and later chooses one of the variables to operate like a "clock". Conditional probabilities are computed for variables of the system to take certain values when the "clock" specifies a certain time. This framework is attractive in contexts where the assumption of usual quantum mechanics of the existence of an external, perfectly classical clock, appears unnatural, as in quantum cosmology. Until recently, there were problems with such constructions in ordinary quantum mechanics with additional difficulties in the context of constrained theories like general relativity. A scheme we recently introduced to consistently discretize general relativity removed such obstacles. Since the clock is now an object subject to quantum fluctuations, the resulting evolution in the time is not exactly unitary and pure states decohere into mixed states. Here we work out in detail the type of decoherence generated, and we find it to be of Lindblad type. This is attractive since it implies that one can have loss of coherence without violating the conservation of energy. We apply the framework to a simple cosmological model to illustrate how a quantitative estimate of the effect could be computed. For most quantum systems it appears to be too small to be observed, although certain macroscopic quantum systems could in the future provide a testing ground for experimental observation.

I. INTRODUCTION

A. Relational time in quantum mechanics

There exists interest in applying the rules of quantum mechanics in situations where there does not exist an obvious way of considering an external perfectly classical clock. An example is provided by quantum cosmology, presumably the correct description of the universe close to the big bang. Sometimes the discussion of quantum cosmology gets further complicated by the problems associated with the quantization of general relativity. But it should be emphasized that the quantization remains troublesome even if one considers simplified model cosmologies, for instance described by Newton's theory (an example is the Barbour–Bertotti model [1]).

Given the lack of an external time, one could try to use a variable internal to the system under study as a clock. Such a variable could not play the role of "t" in a Schrödinger equation, since one expects it to be on a similar footing as the other variables in the problem and therefore should be subject to quantum fluctuations. There have been proposals in the past to build a quantum mechanics using an internal variable as a clock. We will call these proposals "relational time" (see the next subsection for clarifications on terminology). What one does is to promote all variables in the theory to quantum operators, then choose from among the variables one (or several) that will operate as a "clock" and then compute conditional probabilities for the other variables to take certain values when the "clock" variable takes a given value. If one now considers the system as possessing an external perfect classical clock and builds an ordinary Schrödinger description for the system, and if the variable chosen as internal clock behaves semiclassically in such a description, then the relational picture and the Schrödinger picture will give similar descriptions of the system. If the variable chosen as "clock" has large quantum fluctuations, then both descriptions will differ. It is clear that since the "internal clock" will always have some quantum fluctuations, that both descriptions can never be identical. It is therefore of interest to address by how much both descriptions differ. This is one of the main purposes of this paper.

In ordinary quantum mechanics, there is an old argument due to Pauli that one cannot promote a dynamical variable to a quantum operator associated with time, since this operator would have to be canonically conjugate to the Hamiltonian, and since time is expected to be monotonous, this would imply that the Hamiltonian cannot be bounded from below. We will see that in our approach this difficulty is not present.

There is an additional complication. If one considers a system as alone in the universe, without external observers, there appear constraints among the variables of the system. Such systems are called "generally covariant" or "totally constrained". For instance, in the Barbour–Bertotti [1] model, which consists of a few Newtonian particles alone in the universe, one has the constraint that the total momentum, energy and the total angular momentum must have fixed values. In such systems, although one can formally write them as dependent on a parameter called "t", one can

always reparameterize them and there is no natural choice of a variable to identify with a physical time. Sometimes this leads to calling them "reparametrization invariant" systems. In a cosmology based on general relativity, one has the Hamiltonian and momentum constraints. If one formulates the theory canonically in order to quantize it, the constraints have to have vanishing Poisson brackets with any quantity that can be considered physically. This presents problems, since it implies that these quantities are constants of the motion, and cannot be used as "clocks". Page and Wootters [2] attempt to bypass this by choosing to build the relational framework in terms of quantities that do not have vanishing Poisson brackets with the constraints (that is, they choose to work at the kinematical level.) However, quantum mechanically, the states that are annihilated by the constraints are distributional within the space of kinematical states and do not lead to a good probabilistic interpretation. The resulting propagators are proportional to the Delta function and therefore "they don't propagate" as discussed in detail by Kuchař [3].

We therefore see that the presence of constraints interfere with the idea of attempting the introduction of a relational time in quantum mechanics. One possibility to circumvent the problem is to get rid of the constraints. We have recently proposed a discretization of field theories that has the property that when applied to totally constrained systems, the resulting discrete theories are constraint free. In spite of this, they manage in certain circumstances, to approximate well the continuum theories from which they were generated. This allows the introduction of a relational notion of time in the discrete theories. In this paper we would like to discuss, within the context of this approach, what are the quantitative effects of evolving a system using the relational time.

It is clear that the relational time can be introduced in any context that would produce a system that is free of constraints. The discretized approach we consider is an example of such a context, but there could exist others.

B. A note on terminology

The idea of using a given dynamical variable in a closed system as a clock is quite old. One can find traces of it in the work of Leibnitz, Mach, Bergmann, DeWitt, Page, Wootters, Halliwell, Ashtekar, Rovelli, Smolin, Mermin and many others. For a partial list see [2, 3, 4]. The word "relational" is also used in an attractive proposal of Rovelli [5] in which observables (that is, quantities that have vanishing Poisson brackets with the constraints in constrained systems) are constructed via a relational technique. The proposal differs from the one presented in this paper. In it, the observables are constructed as dependent on a clock variable that is not promoted to a quantum operator like we do here.

II. CONSISTENT DISCRETIZATIONS: BRIEF SUMMARY

A. General framework

We summarize here the consistent discretization approach, for a lengthier discussion see [6]. Readers whose main interest is not the details of how to handle general relativity can skip this section. We start by considering a system with an action as in ordinary classical mechanics dependent on a finite number of configuration variables q^a, $S = \int d\tau L(q^a, \dot{q}^a)$. Although this may appear oversimplificatory, recall that when one discretizes a field theory one is really dealing with a mechanical system. This Lagrangian should be considered as a very general Lagrangian, in the sense that it can accommodate first order formulations (in which case the momenta are considered as q's) and systems with explicit constraints in the Lagrangian (in which one treats the Lagrange multipliers as q's such that no $\dot{q}$ appears in the Lagrangian). The formalism can accommodate theories invariant under reparameterizations, like general relativity, for that reason we refer to the evolution parameter as τ.

We now discretize the evolution parameter (in the case of a field theory we also discretize space), and replace the derivatives via $\dot{q}^a = (q_{n+1}^a - q_n^a)/\Delta\tau$ (other discretizations of the derivatives are possible, but would require reworking the formulation). The action will then become a sum $S = \sum_{i=0}^{N} L(q_n, q_{n+1})$ where we have absorbed the interval of discretization into the definition of $L(q, q_{n+1})$. The Lagrange equations are obtained by minimizing the action with respect to each of the q_n^b,

$$\frac{\partial L(q_n^a, q_{n+1}^a)}{\partial q_n^b} + \frac{\partial L(q_{n-1}^a, q_n^a)}{q_n^b} = 0. \tag{1}$$

We next introduce a type 1 canonical transformation that corresponds to the evolution of the system from n to $n+1$ with generating functional $-L(q_n^a, q_{n+1}^a)$,

$$p_{n+1}^b = \frac{\partial L(q_n^a, q_{n+1}^a)}{\partial q_{n+1}^a}, \tag{2}$$

$$p_n^b = -\frac{\partial L(q_n^a, q_{n+1}^a)}{\partial q_n^a}. \tag{3}$$

It is immediate to see, by evaluating the top equation at $n = n - 1$ that the two equations are equivalent to the Lagrange equations. These two equations should be seen as an implicit map between q_n^a, p_n^a and q_{n+1}^a, p_{n+1}^a. If the implicit map is invertible, then the resulting transformation is indeed canonical, by construction.

If the map is not invertible, that is, if the determinant $\det\frac{\partial^2 L}{\partial q_n^a q_{n+1}^b}$ vanishes identically, one has constraints. We denote them generically as $\phi^A(q_n^a, p_n^a)$, $A = 1\ldots M$. The variables at instant $n+1$ are not completely determined by the variables at instant n, and will depend on some free parameters V^B, $B = 1\ldots M$,

$$q_{n+1} = q_{n+1}(q_n, p_n, V^B) \tag{4}$$
$$p_{n+1} = p_{n+1}(q_n, p_n, V^B). \tag{5}$$

It should be noted that the constraints not only occur at level n but at all levels. We will treat them by considering the constraints at level n and then ensuring conservation upon evolution. This can lead to four different situations: 1) imposing the constraints at all levels leads to the determination of some of the indeterminate parameters V^B; 2) new (secondary) constraints appear; 3) the constraints are preserved automatically; 4) the system is incompatible. In the three compatible situations one generically ends up with evolution equations, potentially depending on some free parameters and potentially with extra constraints. It can be shown that the resulting evolution equations [6] preserve the Poisson brackets and (weakly) the constraint surface.

It should be noted that preserving constraints in the discrete theory is harder to accomplish than in the continuum theory. In the latter one only needs to show conservation infinitesimally, and the calculations neglect higher order terms in the (infinitesimal) time interval. In the discrete theory, since the evolution is finite, one needs to show conservation preserving all powers of the time interval. For this reason, in many circumstances constraints that are first class in the continuum become second class when discretized. In our approach, constraints are eliminated by solving them for the Lagrange multipliers. One therefore ends up with a discrete theory that is free of constraints that nevertheless approximates a continuum theory that has constraints. In the next subsection we illustrate this procedure in detail by considering an example in general relativity.

To quantize the theory one has to deal with the constraints that may be left, since as we mentioned it may not be always possible to solve the constraints by choosing the Lagrange multipliers. If the remaining constraints are second class one imposes them strongly and one replaces Poisson brackets by Dirac brackets, if they are first class they are imposed as operatorial identities on the wavefunctions. Finally, one needs to implement the canonical transformation as a unitary evolution operator.

B. A cosmological example

We will consider a Friedmann cosmological model, written in terms of Ashtekar's variables [7]. The fundamental canonical pair is (E, A) where E is the only remnant of the triad after the minisuperspace reduction and A is its canonically conjugate variable. We will consider the presence of a cosmological constant and of a scalar field. We will assume the scalar field has a very large mass so we can neglect its kinetic term in the Hamiltonian constraint, for the sake of computational simplicity. The Lagrangian for the model is,

$$L = E\dot{A} + \pi\dot{\phi} - NE^2(-A^2 + (\Lambda + m^2\phi^2)|E|) \tag{6}$$

where Λ is the cosmological constant, m is the mass of the scalar field ϕ, π is its canonically conjugate momentum and N is the lapse with density weight minus one. The appearance of $|E|$ in the Lagrangian is due to the fact that the term cubic in E is supposed to represent the spatial volume and therefore should be positive definite.

We consider the evolution parameter to be a discrete variable. Then the Lagrangian becomes

$$L(n, n+1) = E_n(A_{n+1} - A_n) + \pi_n(\phi_{n+1} - \phi_n) - N_n E_n^2(-A_n^2 + (\Lambda + m^2\phi_n^2)|E_n|) \tag{7}$$

If one carries out the construction of the previous subsection, one ends up with the following evolution equations, (see [8] for details),

$$P_{n+1}^A = A_n^2 \Theta^{-1} \tag{8}$$
$$A_{n+1} = \frac{3A_n^2 - P_n^A \Theta}{2A_n} \tag{9}$$
$$\phi_{n+1} = \phi_n \tag{10}$$
$$P_{n+1}^\phi = P_n^\phi - \left(A_n^3 - P_n^A \Theta A_n\right) m^2 \phi_n \Theta^{-2} \tag{11}$$

$\Theta = \Lambda + m^2 \phi_n^2$. The variable E and its canonical momentum have been eliminated, and we have solved the constraints of the model for the Lagrange multiplier N. This evolution preserves the Poisson brackets among the canonical pairs (A, P^A) and (ϕ, P^ϕ).

The classical solution of the finite difference equations can be seen to approximate well the continuum solution (see [8] for details). The approximation gets better for later times, and one enters a "continuous regime" in which one can approximate the dynamics by a differential equation in terms of a continuous variable. Such equation can be integrated exactly and from there we can read the behavior of the variables asymptotically into the future. The result is,

$$A = \ell_{\text{Planck}}\sqrt{\Lambda}a(n+k)^{2/3}, \tag{12}$$

$$E = \ell_{\text{Planck}}^2 a^2 (n+k)^{4/3}, \tag{13}$$

where a and k are integration constants, which we chose to be dimensionless and therefore we made explicit the appropriate dimensions through the introduction of the Planck length ℓ_{Planck}. The connection A is therefore dimensionless since the cosmological constant Λ has dimensions of $(\text{length})^{-2}$.

The model can be quantized by implementing the canonical evolution equations as operator equations via a unitary evolution operator,

$$< A', \phi'|U|A, \phi > = \text{sg}(A)\sqrt{\frac{2|A|}{\Theta}}e^{i\text{sg}(A)A^2\frac{A'-A}{\Theta}}\delta(\phi' - \phi) \tag{14}$$

Dirac [9] had already noted in 1933 that the unitary operator that implements a canonical transformation is given by $\exp(-iG)$ where G is the generating function of the canonical transformation. In our case the generating function (after eliminating the Lagrange multipliers is indeed given by $G(A_n, A_{n+1}) = A_n^2(A_{n+1} - A_n)/\Theta$. There is an overall difference with Dirac's result since he chooses a specific factor ordering that does not coincide with the one we chose. It is interesting that this construction is what led Dirac to the notion of path integral.

To define a time we introduce the conditional probabilities, "probability that a given variable have a certain value when the variable chosen as time takes a given value". For instance, taking A as our time variable, let us work out first the probability that the scalar field conjugate momentum be in the range $\Delta P^\phi = [P^\phi_{(1)}, P^\phi_{(2)}]$ and "time" is in the range $\Delta A = [A_{(1)}, A_{(2)}]$ (the need to work with ranges is because we are dealing with continuous variables). Since we have no constraints, the wavefunctions $\Psi[A, \phi, n]$ in the Schrödinger representation admit a probabilistic interpretation. Therefore the probability of simultaneous measurement is,

$$P_{\text{sim}}(\Delta P^\phi, \Delta A) = \lim_{N \to \infty} \frac{1}{N}\sum_{n=0}^{N}\int_{P^\phi_{(1)}, A_{(1)}}^{P^\phi_{(2)}, A_{(2)}} \Psi^2[A, P^\phi, n]dP^\phi dA. \tag{15}$$

We have summed over n since there is no information about the "level" of the discrete theory at which the measurement is performed, since n is just a parameter with no physical meaning. With the normalizations chosen if the integral in P^ϕ and A were in the range $(-\infty, \infty)$, P_{sim} would be equal to one.

To get the conditional probability $P_{\text{cond}}(\Delta P^\phi|\Delta A)$, that is, the probability that having observed A in ΔA we also observe P^ϕ in ΔP^ϕ, we use the standard probabilistic identity

$$P_{\text{sim}}(\Delta P^\phi, \Delta A) = P(\Delta A)P_{\text{cond}}(\Delta P^\phi|\Delta A) \tag{16}$$

where $P(\Delta A)$ is obtained from expression (15) taking the integral on P^ϕ from $(-\infty, \infty)$. We therefore get

$$P_{\text{cond}}(\Delta P^\phi|\Delta A) = \frac{\lim_{N\to\infty}\frac{1}{N}\sum_{n=0}^{N}\int_{P^\phi_{(1)}, A_{(1)}}^{P^\phi_{(2)}, A_{(2)}} \Psi^2[A, P^\phi, n]dP^\phi dA}{\lim_{N\to\infty}\frac{1}{N}\sum_{n=0}^{N}\int_{-\infty, A_{(1)}}^{\infty, A_{(2)}} \Psi^2[A, P^\phi, n]dP^\phi dA}. \tag{17}$$

Notice that all the integrals are well defined and the resulting quantity behaves as a probability in the sense that integrating from $(-\infty, \infty)$ in P^ϕ one gets unity.

We will return to this example when we attempt to estimate the effects of decoherence in our current universe.

III. RELATIONAL TIME IN GENERALLY COVARIANT SYSTEMS

Let us return now to the general discussion of the use of a relational time in generally covariant systems. We assume we have discretized the system of interest using the techniques describe in the previous section. Therefore the

resulting discrete theory is either free of constraints, or if, there are constraints left, we need to ensure we work with physical quantitites that have vanishing Poisson brackets with the remaining constraints. An example of this would be general relativity written in terms of Asthekar variables in a generic situation. There our technique eliminates the diffeomorphism and Hamiltonian constraint but leaves the Gauss law as a constraint. One therefore would have to build the arguments that follow involving quantities that are invariant under gauge transformations, for instance using Wilson loops.

As we discussed in the previous section, we have a unitary operator to be related to the canonical transformation on the phase space $q_n, p_n \to q_{n+1}, p_{n+1}$. We shall denote this operator as $U(n, n')$ for a general displacement $n \to n'$. At this point it is worthwhile re-examining the Pauli argument against promoting time to a quantum operator. Since in the discrete approach the evolution of the system from n to $n+1$ is done through a unitary transformation, corresponding to a discrete canonical transformation in the classical theory, the evolution is not associated to a Hamiltonian. One can define (locally in time) objects that behave close to a Hamiltonian at least when one considers configurations that approximate the continuuum well. But such objects do not exist globally [10]. Therefore one generically does not have a Hamiltonian with which to build the argument laid out by Pauli, which is of global nature. Notice that in terms of the parameter n the evolution is globally defined, but n is not what is promoted to an operator representing time.

Let us introduce now the set of self-adjoint operators $\hat{O}_n(\hat{q}_n, \hat{p}_n)$ with eigenvalues o on the isomorphic set of Hilbert spaces $\mathcal{H}_n = L^2(q_n)$. We can construct now the set of projectors $P_o(n) = \int_{\Delta o} \sum_j |o, j, n><o, j, n|$, where we have assume a continuum spectrum, and denote the eigenvalues of all other operators that form a complete set with $\hat{O}$, as j.

We will construct the conditional probability to measure the value o (since we consider a continuous spectrum, one should strictly speak of measuring a value within an interval Δo surrounding the value o) for the partial observable $O(q_n, p_n)$. We also consider a "clock variable" $T(q, p)$ and the associated self-adjoint operator $\hat{T}_n(\hat{q}_n, \hat{p}_n)$ with eigenvalues t. We define the projector $P_t(n) = \int_{\Delta t} \sum_k |t, k, n><t, k, n|$ as we did for $\hat{O}_n$. In this case k denotes the eigenvalues of all the operators that form a complete set with $\hat{T}$. These are well defined projectors for each n value, that is $P_a(n)^2 = P_a(n), P_a(n)P_{a'}(n) = \delta(a - a')P_a(n), \sum_a P_a(n) = 1 \, \forall \, n$.

We are now ready to introduce the relational time. For an application of these ideas in the simple example of the parameterized non-relativistic particle see [11]. The relational interpretation is defined as follows. Let us assume that the system is initially in the state described by the density matrix ρ. The conditional probability to obtain the value $o \in \Delta o$ for the quantity O given the value $t \in \Delta t$ for T is,

$$\mathcal{P}(o \in \Delta o | t \in \Delta t)_\rho = \frac{\sum_n \text{Tr}(P_o(n)P_t(n)\rho P_t(n))}{\sum_n \text{Tr}(P_t(n)\rho)}. \tag{18}$$

It should be emphasized the important role that the presence of the parameter n plays in these formulas. Without such an ordering parameter, one could not define the conditional probabilities. The parameter n introduces a notion of simultaneity in the construction (at a given spatial point). In fact, several previous attempts to introduce relational times were problematic due to the lack of such a parameter. For instance, this led Unruh [12] to attempt to introduce in an ad-hoc manner a "mysterious time" in the continuuum theory to play such a role.

If we further assume that $\hat{O}$ and $\hat{T}$ commute, we can construct the projector

$$P_{o,t}(n) = \sum_l \int_{\Delta o, \Delta t} |o, t, n, l><o, t, l, n| \tag{19}$$

and rewrite equation (18) as,

$$\mathcal{P}(o \in \Delta o | t \in \Delta t)_\rho = \frac{\sum_n \text{Tr}(P_{o,t}(n)\rho)}{\sum_n \text{Tr}(\int do P_{o,t}(n)\rho)} \tag{20}$$

and from now on for simplicity we will assume that $\hat{O}$ and $\hat{T}$ commute.

Due to the unitary evolution and the cyclic property of the trace, if we define the operators $\Pi_o = \sum_n P_o(n), \Pi_t = \sum_n P_t(n)$ and $\Pi_{o,t} = \sum_n P_{o,t}(n)$, we can rewrite the conditional probability as,

$$\mathcal{P}(o \in \Delta o | t \in \Delta t)_\rho = \frac{\text{Tr}(\Pi_{o,t}\rho)}{\text{Tr}(\int do \Pi_{o,t}\rho)} \equiv \frac{\text{Tr}(\Pi_{o,t}\rho)}{\text{Tr}(\Pi_t \rho)}. \tag{21}$$

The reduction postulate is given by,

$$\rho \to \sum_n P_{o,t}(n)\rho P_{o,t}(n). \tag{22}$$

This reduction process allows us to calculate the conditional probability that defines the correlation functions (propagators),

$$\mathcal{P}(o'|t',o,t,\rho) = \frac{\sum_{n,n'} \text{Tr}(P_{o',t'}(n')P_{o,t}(n)\rho P_{o,t}(n))}{\sum_{n,n'} \text{Tr}(P_{t'}(n')P_{o,t}(T)\rho P_{o,t}(n))}. \tag{23}$$

The difference with usual propagators is that here the times t, t' are the outcome of a quantum measurement. Therefore this will lead to a meaningful definition of probability if we interpret the above expression as having measured o,t and t' and asking what is the probability that one will measure o' given those measurements. Notice that at the moment there is no well defined concept of "time ordering", since the latter is only expected to arise in semiclassical regimes. There is however a well defined notion of simultaneity (at a given spatial point). In this sense the measurements of o and t should be simultaneous and so should be the measurement of t' and o'.

In non-relativistic quantum mechanics one can compute the probability that measurements of position will find the particle at a position x at a series of times $t_1, t_2, ...t_n$. Here one can compute such probabilities through a natural extension of (23). This introduces a reduction postulate in the framework and leads to the history approach for computing a probability for a succession of events. This works if one chooses as clock a robust variable, i.e., such that the information about the variable is not destroyed in the measurements.

From the previous expression, we have that if one prepares a quantum state with eigenvalue o at time t, $\rho_{o,t}$ the probability of it to evolve into a state with eigenvalue o' at time t' can be written as,

$$\mathcal{P}(o \in \Delta o'|t \in \Delta t')_{\rho_{o,t}} = \frac{Tr(\Pi_{o',t'}\rho_{o,t})}{Tr(\Pi_{t'}\rho_{o,t})}. \tag{24}$$

Since up to now we have made almost no assumptions about the nature of the "time" t chosen, it could happen that the same value of t occurs many times upon "evolution" in the parameter n. This eliminates the predictive power of the theory, at least locally in time in the following sense: one could make a definite prediction only upon completing the entire evolution of the system and determining if the variable t takes a given value more than once. Only then one could make sense of the probabilities and predict the probability of a given observable taking a given value at "time t". We will discuss this in detail in a forthcoming paper.

As we shall see in what follows, there is a particular regime where we can concentrate on a particular range of steps. This regime will correspond to considering as a clock a variable that operates semiclassically. When this happens, then the quantum mechanics we constructed will reproduce the results of ordinary quantum mechanics. The agreement will be better the more classically the clock variable behaves.

Another element to be emphasized is that we have used the parameter n to define simultaneity. One has to be careful in the case of general relativity that one has discretized the theory in such a way that the parameter n is associated with Cauchy surfaces of the continuum theory. Otherwise it would not be correct to use the parameter to define a notion of simultaneity. Even this requirement is not enough. In the consistent discretization scheme the lapse is determined dynamically. One cannot rule out situations in which the lapse becomes negative. In such situations "time runs backwards" and one covers the same region of the manifold more than once. These situations in the classical theory should also be avoided in order to obtain a sensible quantum mechanics. It should also be emphasized that the whole issue of the covariance of the formalism is still to be worked out in detail. In particular, since we are here largely concentrating in models without spatial degrees of freedom, it is difficult to even pose the question of how they transform under coordinate transformations. We are taking into account possible time reparameterizations, but not coordinate changes that mix space and time. We are currently studying model systems in which some limited set of symmetries can be implemented and study how the relational approach will mix with covariance, but we will not discuss this in this paper.

IV. SEMICLASSICAL TIME

We now wish to show that the new quantum mechanics we have created reduces to ordinary quantum mechanics if the time variable chosen behaves classically. If one takes into account semiclassical corrections for the time variable, then one will end up with corrections to ordinary quantum mechanics. At this point it is worthwhile mentioning the work of Egusquiza, Garay and Raya [13]. They have studied modifications to quantum mechanics through the use of imperfect clocks. They treat the clocks classically but admit they may have fluctuations in their behavior, perhaps of quantum mechanical origin, and they model them through a Markovian process with a given probability distribution. There are many points of contact between their calculations and the ones we present here. The main difference relies on the fact that we are treating the clock as quantum mechanical and in particular we allow its probability to evolve

as a function of time. We are not including any thermal or other effect, but these could definitely be incorporated if needed through the formalism of Egusquiza et al.

Let us assume now the existence of a semiclassical regime for the variable chosen as time for a given initial state of the complete system ρ. That means that there exists a region R in the spectrum of the operators $\hat{T}_n$ such that for a value $t \in R$ there is an interval of values $\Delta_t n$ of the step parameter n such that $\mathcal{P}_n(t) \approx 0 \forall n \notin \Delta_t n$. Also given a value of the step parameter n_0, there exists an interval around it $\Delta_{n_0} t$ such that $\mathcal{P}_{n_0}(\tilde{t}) \approx 0 \; \forall \; \tilde{t} \notin \Delta_{n_0} t = 0$. In these expressions $\mathcal{P}_n(t)$ is the probability density of having the value t at a given n. We speak of probability density since we are assuming that the operator T has a continuous spectrum. If the spectrum were discrete then $\mathcal{P}_n(t)$ would be the probability of having the value t at the level n. This semiclassical limit therefore implies a strong correlation between the n parameter and the eigenvalues of the observable T in region R. Notice that in this case each reduction process is almost a projection given by,

$$\rho \to \sum_{n \in \Delta_t n} P_{o,t}(n)\rho P_{o,t}(n). \tag{25}$$

We need to assume also that the clock and the system in study are weakly interacting (otherwise one would not have a reasonable clock). Concretely, we assume that the state of the system is of the form of a product $\rho \approx \rho_1 \bigotimes \rho_2$ throughout the evolution determined by a unitary operator $U \approx U_1 \bigotimes U_2$ also of product form. We take "1" to describe the clock and "2" the physical system under study and the weak interaction also implies that U_1 and U_2 commute.

Up to now we have considered the quantum states as described by a density matrix at a given level of the discretization parameter n. Since the latter is not an observable, we would like to transition to a description where we have density matrices function of the observable time instead of n. In order to do this, let us recall the expression for the usual probability in the Schrödinger representation of measuring the value o at time t in ordinary quantum mechanics,

$$\mathcal{P}(o|t)_\rho \equiv \frac{\text{Tr}(P_o(0)\tilde{\rho}(t))}{\text{Tr}(\tilde{\rho}(t))}, \tag{26}$$

where the projector is evaluated at $t = 0$ since in the Schrödinger representation the operators do not evolve. We would like to obtain an expression similar to this one in the relational time picture. We are denoting $\tilde{\rho}(t)$ as the density matrix in the picture in which we have a time variable, we will drop the tilde to denote the density matrix that arises under the evolution of the step parameter n and we will drop the n dependence to refer to the initial density matrix $\rho(n = 0)$.

We start by considering the conditional probability defined in (20),

$$\mathcal{P}(o \in \Delta o|t \in \Delta t)_\rho = \frac{\sum_n \text{Tr}\left(P_o(n)P_t(n)\rho P_t(n)\right)}{\sum_n \text{Tr}\left(P_t(n)\rho\right)}, \tag{27}$$

and one could have omitted the last $P_t(n)$ using the cyclicity of the trace since we are assuming that P_t and P_o commute. Given that we are assuming the presence of continuous spectra, the projectors in the above expression should be understood as integrated over the interval, i.e. $P_o(n) = \int_{\Delta o} P_{o'}(n)do'$ and similar for P_t. We now introduce the hypothesis that the clock and the rest of the system interact weakly and write explicitly the evolution of the projectors in the step parameter n to get,

$$
\begin{aligned}
\mathcal{P}(o \in \Delta o|t \in \Delta t)_\rho &= \frac{\sum_n \text{Tr}\left(U_2^\dagger(n)P_o(0)U_2(n)U_1^\dagger(n)P_t(0)U_1(n)\rho_1 \otimes \rho_2\right)}{\sum_n \text{Tr}\left(P_t(n)\rho_1\right)\text{Tr}\left(\rho_2\right)} \\
&= \frac{\sum_n \text{Tr}\left(U_2^\dagger(n)P_o(0)U_2(n)\rho_2\right)\text{Tr}\left(U_1^\dagger(n)P_t(0)U_1(n)\rho_1\right)}{\sum_n \text{Tr}\left(P_t(n)\rho_1\right)\text{Tr}\left(\rho_2\right)}.
\end{aligned}
\tag{28}
$$

From this expression, using the cyclic property of the trace, we can identify the expressions of the density matrix evolved in relational time. We start by defining the probability that the measurement t corresponds to the value n,

$$\mathcal{P}_n(t) \equiv \frac{\text{Tr}\left(P_t(0)U_1(n)\rho_1 U_1^\dagger(n)\right)}{\sum_n \text{Tr}\left(P_t(n)\rho_1\right)}, \tag{29}$$

and notice that $\sum_n \mathcal{P}_n(t) = 1$.

We now define the evolution of the density matrix,

$$\tilde{\rho}_2(t) \equiv \sum_n U_2(n)\rho_2 U_2^\dagger(n)\mathcal{P}_n(t), \tag{30}$$

and noting that

$$\mathrm{Tr}\left(\tilde{\rho}_2(t)\right) = \sum_n \mathcal{P}_n(t)\mathrm{Tr}\left(\rho_2\right) = \mathrm{Tr}(\rho_2) \tag{31}$$

one can equate the conditional probability (28) with the usual expression for a probability in quantum mechanics (26). It should be noted that all the sums in n, due to the assumption that the time variable is semiclassical are only nontrivial in the interval $\Delta_t n$ since outside of it, probabilities vanish. Something else to notice is that when we introduced the projectors, there was an integral over an interval. Therefore in the above expression for the evolution of the density matrix, this has to be taken into account. Since the interval Δt is arbitrary, one can consider the limit in which its width tends to zero, apply the mean value theorem in the integrals, and the interval in the numerator and denominator cancel out, yielding an expression for $\tilde{\rho}_2(t)$ that is independent of the interval, and involves the non-integrated projector $P_t(0)$.

We have therefore ended with the standard probability expression with an "effective" density matrix in the Schrödinger picture given by $\tilde{\rho}_2(t)$. In its definition, it is evident that unitarity is lost, since one ends up with a statistical mixture of states associated with different n's. We also notice that probabilities are conserved, as can be seen by taking (26) and integrating over x. We recall that $\tilde{\rho}_2$ is not the normalized density matrix; the latter can be easily recovered dividing by the trace.

We will assume that $\mathcal{P}_n(t) \equiv f(t - t_{max}(n))$ with f a function that decays quite rapidly for values of t distant of the maximum $t_{\max}$ which depends on n.

To manipulate expression (30) more clearly, we will assume we are considering a finite region of evolution and we are in the limit in which the number of steps in that region is very large. We denote the interval in the step variable n as going from zero to N with N a very large number. We define a new variable $v = \epsilon n$ with dimensions of time such that $N\epsilon = V$ with V a chosen finite value. We can then approximate expression (30) by a continuous expression,

$$\tilde{\rho}_2(t) = \int_0^V dv f(t - t_{max}(v))\rho_2(v). \tag{32}$$

In this expression $t_{\max}(v) \equiv t_{\max}(n = v/\epsilon)$ and

$$\rho_2(v) = U_2(n = v/\epsilon)\rho_2 U_2^\dagger(n = v/\epsilon). \tag{33}$$

In all the above expressions, when we equate $n = v/\epsilon$ it should be understood as $n = \mathrm{Int}(v/\epsilon)$, which coincide in the continuum limit. (Notice that strictly speaking we should write $\rho_2(v/\epsilon)$ to keep the same functional form as for $\rho_2(n)$, but we will drop the ϵ to simplify the notation.)

Let us assume now that $t_{max}(v) = v + \epsilon\Gamma_1(v) + \epsilon^2\Gamma_2(v)$, with $\Gamma_1(v) \sim 1$ and $\Gamma_2(v) \sim 1$, that is, the value at which $\mathcal{P}_n(t)$ has its maximum depends linearly on n plus a small correction. One can always obtain the classical dependence of t on n solving the equations of the theory. The linear approximation can be justified locally as a Taylor expansion, or by redefining the variable chosen as time such that the relation with n is linear.

To simplify calculations we replace the function f with a function $\tilde{f}$ which agrees with it up to terms of order ϵ^2,

$$f(t - t_{\max}(v)) = f(t - v - \epsilon\Gamma_1(v) - \epsilon^2\Gamma_2(v)) \equiv \tilde{f}(v - t + \epsilon\Gamma_1(t) + \epsilon^2\Gamma_2(t) - \epsilon^2\alpha\Gamma_1(t)), \tag{34}$$

where $\alpha(v) \equiv \partial\Gamma_1(v)/\partial v$ and we have neglected terms of order ϵ^3.

We now assume that f is well approximated by a Dirac delta,

$$\begin{aligned}
\tilde{f}(v - t + \epsilon\Gamma_1(t) + \epsilon^2\Gamma_2(t) - \epsilon^2\alpha\Gamma_1(t)) &= c_0(t)\delta(v - t + \epsilon\Gamma_1(t) + \epsilon^2\Gamma_2(t) - \epsilon^2\alpha\Gamma_1(t)) \\
&+ a(t)\epsilon\delta'(v - t + \epsilon\Gamma_1(t)) + b(t)\epsilon^2\delta''(v - t) + O(\epsilon^3)
\end{aligned} \tag{35}$$

with $b > 0$. Since $\mathcal{P}_v(t) = \tilde{f}(v - t + \epsilon\Gamma_1(t) + \epsilon^2\Gamma_2(t) - \epsilon^2\alpha\Gamma_1(t))$ and since we want $\int_0^V \mathcal{P}_v(t) = 1$ then $c_0(t) = 1$.

We now substitute the explicit form of $\tilde{f}$ (35) in (32), which allows to compute the integral in v explicitly,

$$\tilde{\rho}_2(t) = \rho_2(t - \epsilon\Gamma_1(t) - \epsilon^2\Gamma_2(t) + \epsilon^2\alpha(t)\Gamma_1(t)) - a(t)\epsilon\frac{\partial}{\partial v}\rho_2(t - \epsilon\Gamma_1(t)) + b(t)\epsilon^2\frac{\partial^2}{\partial v^2}\rho_2(t) + O(\epsilon^3). \tag{36}$$

We proceed to expand the arguments of ρ_2 in this expression in powers of ϵ. We also use the fact that in the continuum limit we can now associate a Hamiltonian with the evolution operator $U_2(v) = \exp(iH_2 v)$, and recalling that the evolution of the density matrix is $\rho_2(v) = U_2(v)\rho_2 U_2^\dagger(v)$, we get,

$$\tilde{\rho}_2(t) = \rho_2(t) + (i\epsilon(\Gamma_1(t) + a(t)))[H_2, \rho_2(t)] - i\epsilon^2 (\alpha(t)\Gamma_1(t) - \Gamma_2(t)) [H_2, \rho_2(t)]$$
$$-\epsilon^2(\frac{1}{2}\Gamma_1(t)^2 + a(t)\Gamma_1(t) + b(t))[H_2, [H_2, \rho_2(t)]]. \tag{37}$$

Notice that generically, one can write the discrete generator as $U_2(n) = \exp(iH_2^{\mathrm{dis}}n)$. However, there may be points where the canonical transformation is singular, and example is the Big Bang in the cosmological model [10], the series defining the logarithnm of U_2 fails to converge and therefore H_2 does not exist. Notice also that in taking the continuum limit $H_2^{\mathrm{dis}}/\epsilon = H_2$.

We will find useful to have the inverse relation between the density matrices,

$$\rho_2(t) = \tilde{\rho}_2(t) - (i\epsilon(\Gamma_1(t) + a(t)))[H_2, \tilde{\rho}_2(t)] - \epsilon^2(\Gamma_1(t) + a(t))^2[H_2, [H_2, \tilde{\rho}_2(t)]] \tag{38}$$
$$+i\epsilon^2 (\alpha(t)\Gamma_1(t) - \Gamma_2(t)) [H_2, \tilde{\rho}_2(t)] + \epsilon^2 \left(\frac{1}{2}\Gamma_1(t)^2 + a(t)\Gamma_1(t) + b(t)\right) [H_2, [H_2, \tilde{\rho}_2(t)]].$$

We now take the time derivative of equation (37) and we note that the derivatives will act on the coefficients $a, b, \alpha, \Gamma(t)$ or on $\rho_2(t)$. We replace $\dot{\rho}_2(t)$ with the commutator with the Hamiltonian. The final result, using (38) to rewrite everything in terms of $\tilde{\rho}_2$ is

$$\frac{\partial\tilde{\rho}_2(t)}{\partial t} = i\left(-1 + \epsilon\frac{\partial}{\partial t}(\Gamma_1(t) + a(t)) - \epsilon^2\frac{\partial(\alpha(t)\Gamma_1(t) - \Gamma_2(t))}{\partial t}\right)[H_2, \tilde{\rho}_2(t)] + \epsilon^2\left[\frac{\partial}{\partial t}\left(\frac{a(t)^2}{2} - b(t)\right)\right][H_2, [H_2, \tilde{\rho}_2(t)]]. \tag{39}$$

The general form of the resulting evolution equation for the system under study is therefore,

$$\frac{\partial}{\partial t}\tilde{\rho}_2 = -i[(1 + \beta(t))H_2, \tilde{\rho}_2] - \sigma(t)[H_2, [H_2, \tilde{\rho}_2]], \tag{40}$$

with the coefficients $\beta(t)$ and $\sigma(t)$ that are functions that are small corrections (in terms of the expansion in ϵ), and we have neglected terms involving triple and higher commutators with the Hamiltonian since they correspond to higher powers of ϵ

The presence of $\beta(t)$ just redefines the Hamiltonian of the theory. The coefficient $\sigma(t)$ is the one that gives rise to new effects. In particular it will imply that pure states evolve into mixed states. In fact, the equation we have obtained is a particular case of equations that are considered in the context of decoherence in quantum mechanics, called Lindblad [14] type equations

$$\frac{\partial\rho}{\partial t} = -i[H, \rho] - \mathcal{D}(\rho), \tag{41}$$

with $\mathcal{D}(\rho)$ satisfying the properties,

$$\mathcal{D}(\rho) = \sum_n [D_n, [D_n, \rho]], \quad D_n = D_n^\dagger, \quad [D_n, H] = 0, \tag{42}$$

so it defines a completely positive map on ρ that is consistent with the monotonous increase of Von Neumann entropy $S = \mathrm{Tr}(\rho \log \rho)$ and conservation of energy. This type of equation was introduced by Ghirardi, Rimini and Weber (GRW) [15] with the aim of providing an objective solution to the measurement problem in standard quantum mechanics. (Similar equations can be used to describe the decoherence due to interaction with an environment, see [16].) GRW considered a single D as a localizing operator in coordinate space. As discussed by Adler and Horwitz [17], and also Milburn, Percival and Hughston [18, 19], setting D to be proportional to H is most natural since it leads to an objective state vector reduction in the energy pointer basis. This loss of coherence may be a way to avoid macroscopic superpositions, like the "Schrödinger cat" [17, 18].

We can recognize that in our case there is only one D_n that is given by the Hamiltonian. Having a Lindblad form is desirable since it implies that the Hamiltonian will be conserved automatically by the evolution considered. Other proposals for decoherence from quantum gravity, like Hawking's $-matrix may have problems with the conservation of energy [20].

In order to study the influence of $\beta(t)$ and $\sigma(t)$, we start by assuming that they are constant as functions of time $\beta(t) = \bar{\beta}$ $\sigma(t) = \bar{\sigma}$. In general these functions will be constant will some small fluctuations, in which case we can interpret the bars as average values. In such case the equation can be solved exactly,

$$\rho_{2nm}(t) = \rho_{2nm}(0)e^{-i(1+\bar{\beta})\omega_{nm}t}e^{(-\bar{\sigma}(\omega_{nm})^2)t}, \tag{43}$$

where we have written the density matrix element in the basis of energy eigenstates, and ω_{nm} is the Bohr transition frequency between the state n and m in the energy basis. It is important to notice that $\bar{\sigma}$ has to be positive for the evolution to be physically acceptable. Otherwise the trace of the square of the density matrix will be larger than one.

Let us estimate the sign of

$$\sigma(t) = -\epsilon^2 \left[\frac{\partial}{\partial t} \left(\frac{a(t)^2}{2} - b(t) \right) \right]. \tag{44}$$

The coefficient $a(t)$ represents the asymmetry of the probability of the time t as a function of the evolution parameter n. There is no reason for this probability distribution to have a definitive asymmetry, so its mean value through the evolution will vanish. The coefficient $b(t)$ represents the spread of the probability distribution. Suppose one starts the clock in a quantum state that is peaked around a certain value t_0. As the parameter n evolves, the quantum state will disperse and the spread of the probability of t as a function of n will disperse too. Therefore $b(t)$ will increase and its derivative will be positive. The magnitude of the effect will depend on the details of the clock. For instance, if one used a free particle as clock the spread of the wavepacket will be linear and therefore $\sigma(t)$ will be a constant. The important point is that it is positive, and therefore signals an arrow of time.

In the real world, the subsystem chosen as a clock will be subject to several other kinds of fluctuations, for instance thermal fluctuations. The influence of these kinds of errors leads [13] to an equation similar to (40), but where the $\sigma(t)$ is proportional to α^2. We therefore see that a quantum evolution will exhibit decoherence from fundamental effects, like the ones considered in this paper, and also environmental effects. The latter could in some cases be minimized by choosing the experimental setup in appropriate ways, the fundamental effects will, however, be an ultimate limit to the achievable coherence in a quantum evolution.

It should be noted that the value of σ depends the choice of clock, and on the quantum state chosen for the clock initially, and on the interaction of the clock with the environment.

V.   ESTIMATING THE MAGNITUDE OF THE DECOHERENCE

We will now proceed to use the cosmology we discussed earlier to give an estimate for the level of decoherence. This can only be viewed as a first calculation, a more realistic model would be desirable. In particular, we have chosen a model with a scalar field in the infinite mass limit, in which the model is a DeSitter universe. This is not a very good model for approximating the universe since it is homogeneous in time (and in particular the "Big Bang" we refer to is just a coordinate singularity). One should think of it only as asymptotically approximating our universe into the future. It should only be taken as a guide of the way to compute σ. The calculations could be repeated in a more realistic model, but would be more involved.

The task at hand is to provide a justification for the lattice spacing of the discrete model. It is evident that such spacing could not be determined if we just considered a non-gravitational system as our only object of study. Considering the influence of gravity will allow us to introduce a fundamental length scale in the problem, the Planck scale.

Recalling the definition of b in (35) we can relate it to the quadratic deviation of the time variable from its linear behavior $t = \epsilon n$, due to the spread as a function of time of the wavepacket representing the clock through

$$(\Delta t)^2 \equiv \epsilon^2 b = \frac{(\Delta\epsilon)^2}{\epsilon^2} n^2 \epsilon^2 = \frac{(\Delta\epsilon)^2}{\epsilon^2} t^2 \tag{45}$$

therefore we can estimate σ to be,

$$\sigma = \epsilon^2 \dot{b} \sim \frac{(\Delta\epsilon)^2}{\epsilon^2} t = \frac{(\Delta\epsilon)^2}{\epsilon} n \tag{46}$$

(we will neglect factors of order one in working out the estimates).

We now need to choose a time variable in the cosmology. As before, we will choose it related to the connection A, but we also wish to have a linear relation between time and n, as we assumed in the calculation of the decoherence. Therefore we rescale

$$t = \frac{1}{\Lambda^{3/4} \ell_{\text{Planck}}^{1/2}} A^{3/2}, \tag{47}$$

which taking into account the asymptotic behavior of the cosmological variables we described in section (II B) yields,

$$t = a^{3/2} \ell_{\text{Planck}} (n + k) \tag{48}$$

from which we can therefore read off $\epsilon = a^{3/2}\ell_{\text{Planck}}$ and therefore $\Delta\epsilon = \frac{3}{2}\sqrt{a}\Delta a\ell_{\text{Planck}}$. We are choosing the integration constant k to be very small compared to n, since we are assuming we are in the asymptotic future region. It should be noted that the singularity happens for $n + k \sim 0$.

So we can therefore estimate σ,

$$\sigma = \frac{\ell_{\text{Planck}}^2 a(\Delta a)^2}{\epsilon} n. \tag{49}$$

We can now proceed to find a bound on $a(\Delta a)^2$ based on the uncertainty principle, $\Delta E \Delta A > \ell_{\text{Planck}}^2$, which yields,

$$a(\Delta a)^2 > \frac{1}{\ell_{\text{Planck}}\sqrt{\Lambda}n^2} \tag{50}$$

so we finally get for the decoherence parameter,

$$\sigma > \frac{\ell_{\text{Planck}}}{\epsilon\sqrt{\Lambda}n} = \frac{\ell_{\text{Planck}}}{\sqrt{\Lambda}t}. \tag{51}$$

At this point it is worthwhile pondering what is meant in this equation by "t". We have identified this variable as related linearly to the parameter n via equation (47). This choice involves selecting a prefactor with dimensions of time. This prefactor is arbitrary. Therefore it would appear that the result for σ is dependent on this factor. Indeed it is. However, it should be noted that a similar factor arises in identifying the continuum limit Hamiltonian H_2 from the discrete one H_2^{dis} as we discussed before. Therefore if one defines a new time $t' = \lambda t$, the evolution equation for the density matrix acquires an overall factor $1/\lambda$. This requires rescaling σ in the term involving two Hamiltonians by a factor of λ to compensate for the fact that the Hamiltonian is rescaled by $1/\lambda$. Therefore although the value of σ has this arbitrariness, the speed at which states decohere is the same, just measured in different time units.

We still have to ask ourselves how do we connect the time t appearing in the formula for σ and the physical time one would measure in a laboratory experiment. We notice that there exists a relation between t and t_{lab} of the form $t = t(A(t_{\text{lab}}))$. That is, t is connected with the variable we chose as clock A, and the latter is a physical variable of the cosmology that should be connected with the clock in the lab (just like one measures the age of the universe in seconds as defined by a regular clock). This will rescale the left hand side of the evolution equation of the density matrix of the form

$$\frac{\partial t}{\partial t_{\text{lab}}} \frac{\partial \rho}{\partial t} \tag{52}$$

The prefactor $\partial t/\partial t_{\text{lab}}$ will be slowly varying with respect to the time scales of a laboratory experiment. Therefore we can take it to be a constant λ of the same nature as the one we discussed in the previous paragraph. This leads to a definition of the σ one would measure in a lab,

$$\sigma_{\text{lab}} = \frac{\partial t_{\text{lab}}}{\partial t}\sigma. \tag{53}$$

In the model we are discussing, the closest thing to a "physical" time could be the time of comoving observers. This time has an exponential relation to the variable t we have been using. Unfortunately, due to the simplicity of this model there is an overall scale ambiguity and the relation between the comoving time and t is defined up to an overall constant κ. The relation is $t = \kappa\exp(3\sqrt{\Lambda}t/2)$. Therefore the result for the physical σ is,

$$\sigma > \frac{\ell_{\text{Planck}}}{\Lambda\kappa^2\exp(3\sqrt{\Lambda}t_{lab})}. \tag{54}$$

If one were dealing with a universe that is not exactly DeSitter but with a Friedmann model with a power law behavior in comoving time, the relation between comoving time and t would not be exponential but be of the form $\sqrt{\Lambda}t = (\sqrt{\Lambda}t_{\text{lab}})^\alpha$ with α some power, since the Big Bang would have to happen at $t = t_{\text{lab}} = 0$. If that is the case and we assume one is making an experiment today, we have that $\sqrt{\Lambda}t_{\text{lab}} \sim 1$ and therefore (we are setting the speed of light $c = 1$) one has that $\sigma \sim t_{\text{Planck}}$. As expected, the decoherence effect due to the finiteness of the (space)-time lattice is of the order of the Planck scale.

VI. CONCLUSIONS

We have shown how the use of a relational time in quantum mechanics leads to a modification of ordinary quantum mechanics in which pure states evolve to mixed states. The use of a relational time is now possible in quantum gravity due to the use of the consistent discretization framework. One can then proceed to work out estimates of the magnitude of fundamental decoherence due to the fact that in a cosmology time has necessarily to be relational. We have worked out such estimate in this paper using a simple cosmological model. It is clear that more detailed models will be needed to gain confidence in the estimate obtained.

What are the possibilities of detecting the decoherence due to quantum gravity? We have found that the largest amount of decoherence one can expect in a system goes as

$$(\omega_1 - \omega_2)^2 t_{\text{Planck}} t_{\text{life}} \tag{55}$$

where the ω's are the Bohr frequencies corresponding to the two most separated energy levels of the system and t_{life} is how long we wait for the decoherence to appear. For ordinary quantum systems such effect is very small. It is interesting to notice however, that a small level of fundamental decoherence has always been desirable in quantum mechanics to avoid "Schrödinger cat" type situations, and it had been advocated through ad-hoc proposals like that of Ghirardi, Rimini and Weber [15] and others. The most promising experiments where one could observe this effect are the ones involving macroscopic quantum systems, like Bose–Einstein condensates. We have discussed some of these possibilities in reference [21], but it appears detectability is not within reach of current technology, largely because the systems involve a limited number of atoms. Since the energy spread goes as the number of atoms squared, it might be possible that in a few years experiments could reach the desired levels of energy spread in macroscopic quantum systems. It remains to be seen if the specific experimental setups will allow the effect to be sufficiently isolated from other sources of decoherence due to environmental effects.

It should be noted that we are claiming that the consistent discretizations allow to solve the problem of time in quantum gravity and totally constrained systems in general through the introduction of a relational time. The problem of time has many aspects to it. One of these aspects is the identification of a variable that behaves as a "good time". In particular, a variable that is "transverse" to the dynamical orbits of the system. Hájíček et al. [22], Hartle [23] and Kuchař [3] have emphasized that in many cosmological situations there do not exist time variables transverse to the dynamical orbits. Our approach actually addresses this problem. In a nutshell, even in such cases the parameter n in the discretization is transverse to the orbits and this is enough to define a correct relational time. This in particular implies that our approach solves the "time of arrival" problem in quantum mechanics. We will expand on these issues in a forthcoming publication.

Another aspect of the framework that requires further analysis is the issue of the covariance of the predictions. As worked out in this paper, the predictions are particular to a given choice of time. This is unsatisfactory since one should expect the theory at least to exhibit Lorentz invariance locally since it is derived from general relativity. Establishing this in general relativity is however, delicate, since it requires analyzing situations with spatial (field theoretic) degrees of freedom like Gowdy cosmologies which are considerably more involved computationally. We are studying these but we do not have results to report at present. An alternative would be to consider model situations, for instance, two parameterized relativistic particles with a relativistic interaction, but this also requires further study.

Summarizing, the use of consistent discretizations of general relativity free the theory from constraints and therefore one can introduce a relational time as proposed by Page and Wootters avoiding the main objections to such approach. The resulting modified quantum mechanics implies that pure states evolve into mixed states. We have computed explicitly the rate for such decoherence. It may play a role in foundations of quantum mechanics, in solving the black hole information problem [24], and may be observed experimentally.

VII. ACKNOWLEDGMENTS

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